THE SYNTHESIS OF THE ELEMENTS FROM HYDROGEN *

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(Received 1946 April 6 †)

Summary

Stars that have exhausted their supply of hydrogen in regions where thermonuclear reactions are important enter a collapsing phase. If the mass of the star exceeds Chandrasekhar's limit collapse will continue until rotational instability occurs. Rotational instability enables the star to throw material off to infinity. This process continues until the mass of the remaining stellar nucleus becomes of the order of, or less than Chandrasekhar's limit. The nucleus can then attain a white dwarf equilibrium state.

The temperature generated at the centre of a collapsing star is considered and it is shown that values sufficiently high for statistical equilibrium to exist between the elements must occur. The relative abundances of the elements can then be worked out from the equations of statistical mechanics. These equations are considered in detail and it is shown that a roughly uniform abundance of the elements over the whole of the periodic table can be obtained. The process of rotational instability enables the heavy elements built up in collapsing stars to be distributed in interstellar space.

The results arising from the discussion of the formation of heavy elements lead to a natural explanation of the difference between novae and supernovae.

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1. *Introduction*

1.1. *Preliminary remarks.*—The rate at which nuclear re ctions take place in a thermodynamic assembly of material particles and radiation is negligible at ordinary temperatures, except in the case of radioactive nuclei. But as the temperature T increases, the thermal energy of the particles increases until for temperatures in the neighbourhood of 5×10^9 °C. appreciable penetration of the Coulomb barriers of the nuclei by protons and α-particles occurs. Thus, provided the density of the particles is high enough for collisions to be sufficiently frequent the rate of nuclear reactions becomes important at very high temperatures. Indeed for sufficiently large ρ, T the interchanges between nuclei of different atomic weights and nuclear charges become so rapid that statistical equilibrium is set up between the nuclei. The relative abundances of the various elements can then be calculated in terms of ρ, T from the equations of statistical mechanics. Moreover the relative abundances so calculated are independent of the initial composition of the assembly. Thus if we start with an assembly of hydrogen at low temperature and ρ, T are increased until statistical equilibrium occurs the relative abundance will be independent of starting with hydrogen and would have been the same if, for example, we had started with an assembly of helium. The work of Section 5 shows that statistical equilibrium will be set up in a time interval of less than 100 seconds when $\rho = 10^7$ gm. per cm.³ and $T = 4 \times 10^9$ °C. This result is very sensitive to the value of T but not to the value of ρ.

* The writer is indebted to a referee for valuable suggestions concerning the preparation of this paper.

† Received in present form 1946 December 2.

No attempt will be made in the present paper to discuss non-statistical conditions of synthesis of the elements from hydrogen. It is, however, of interest to consider whether this restriction will affect the generality of the conclusions reached. In the writer's opinion no such restriction can arise. There are two arguments that support this point of view. Firstly in order to synthesize the heavy elements on the scale required to explain the occurrence of the heavy elements in the Earth it is necessary that a large number of interchanges between atomic nuclei shall take place, and that these interchanges take place over the whole range of atomic weights. This, however, is essentially the condition required for statistical equilibrium. Stated otherwise, the present argument means that if an appreciable fraction of the hydrogen is to be converted into heavy elements then statistical equilibrium must occur (at any rate approximately) since the number of nuclear reactions necessary to bring about the synthesis must be comparable with, or greater than, the number of particles involved (if only a minute fraction of the hydrogen were converted into heavy elements then non-statistical processes such as occur in laboratory nuclear physics could take place). In addition to this general argument there is an important special argument. In order to synthesize the naturally radioactive elements it is necessary to consider reactions that are the reverse of the radioactive disintegrations. Under statistical equilibrium no special difficulty arises, but under non-statistical conditions it seems impossible to proceed (in a reverse direction) along the radioactive series beyond such extremely short-lived nuclei as *Ra C′* and *Th C′*.

The above remarks suggest the problem of finding values of ρ, T such that the statistical equations give the relative abundances of the elements observed in nature. The phrase "in nature" requires some discussion. The information available concerning the distribution of the elements may be summarized as follows:—

(1) It is known that a large quantity of material occurs both in the Galaxy and in the extragalactic nebulae in the form of a diffuse interstellar gas. Although the amount of this material is not yet accurately determined, there is important observational evidence in favour of the view that the total mass of interstellar material in a nebula exceeds the combined masses of the stars by an appreciable factor. The interstellar material is believed to consist very largely of hydrogen, the quantity of heavy elements amounting to only a few per cent. at most. The heavy elements are present partly in gaseous form and partly as interstellar dust.

(2) As a result of a fairly thorough analysis of the structure of the stars it is now known that a large fraction of the combined masses of the stars is due to hydrogen. In certain stars hydrogen contributes at least 70 per cent., whereas at the other end of the scale stars are known to occur (see Section 2) in which hydrogen is almost entirely absent. This diversity in hydrogen content is due to the interplay of two processes with opposing tendencies:

(i) The generation of energy within a star due to thermonuclear reactions has the net effect of synthesizing helium from hydrogen (in stars of mass less than about half the solar mass $M_\odot$ the helium is commonly believed to be synthesized directly from the hydrogen, whereas in stars of mass greater than about $M_\odot/2$ the synthesis occurs through the carbon-nitrogen cycle). Thus the effect of nuclear reactions *within a normal star* is to increase the helium concentration at the expense of the hydrogen (the distinction between a normal star and a collapsing star is discussed in detail in Section 2). The synthesis from hydrogen of elements heavier than helium is negligible in normal stars * in which the central temperatures never appreciably exceed 5×10^7 °C.

(ii) The loss of hydrogen by synthesis of helium is offset in some stars by rapid accretion of interstellar hydrogen.†

Accordingly we form the following picture of the development of a star during the normal phase of its history.‡ The star condenses from the diffuse interstellar material § and has initially the same composition as the interstellar material. As a result of thermonuclear reactions hydrogen is converted into helium. In some stars the loss of hydrogen is offset by a rapid accretion of further quantities of interstellar material.

(3) Additional data concerning the composition of the stars is obtained from the study of stellar spectra. Although the identification of elements over the whole periodic table is not yet completed, the results so far obtained provide strong confirmation of the view that all elements known on the Earth are also present in the stars. It has been estimated by Russell || that in the surface layers of the stars hydrogen is at least a thousand times as abundant as all the metals combined.

(4) By far the most detailed evidence concerning the distribution of heavy elements comes from the direct study of the contents of the earth's crust.¶ The distribution so obtained differs markedly from the astrophysical evidence discussed above in that only a small quantity of hydrogen and an almost negligible amount of helium is present on the Earth. The most common elements in the crust are silicon, magnesium, and oxygen.

In the present paper we are not so much concerned with the abundance of isolated elements as with suitably averaged values. We may group the elements into a number of ranges of atomic weight, e. g. the ranges 1 to 14 inclusive, 15 to 24 inclusive etc., and average the abundance of all the elements falling within each group. When this is done an averaged abundance is obtained for each range of atomic weight. These averaged abundances rarely fall below one part in a million (uranium forms about one part in 10^5 and lead about five parts in 10^6, so that this statement holds up to the heaviest elements in the periodic table). Now although a difference in abundance between two elements by a factor of 10^3 may be of very great economic importance (one of the elements being regarded as "common" and the other as "rare") such a difference is unimportant from the standpoint of the present paper since factors arise in the present discussion that may be as large as 10^{200}. From the work given below it will appear that a remarkable feature of the survey of elements present in the Earth's crust is the very small variation in abundance that occurs over the whole range of atomic weights.

The above remarks concerning the uniformity of distribution of the elements in the Earth's crust are strengthened by noticing that the heaviest elements would

* H. A. Bethe, *Phys. Rev.*, **55**, 434, 1939.
† F. Hoyle and R. A. Lyttleton, *M.N.*, **101**, 227, 1941.
‡ See Section 2.
§ F. Hoyle, *M.N.*, **105**, 302, 1945.
|| H. N. Russell, *Ap. J.*, **78**, 239, 1933.
¶ V. M. Goldschmidt, *Skr. norske Vidensk Akad.*, 1937.

sink towards the centre when the Earth was in a liquid state. Thus the proportion of elements with atomic weights greater than 40 in the Earth as a whole may well be considerably higher than is indicated by a study of the crust alone. Results obtained from observations of the propagation of sound waves through the body of the Earth confirm this statement. It is found* that the density of material near the centre of the Earth is close to twelve. Allowing for increase of density arising from the high pressure in the inner regions of the Earth it has been estimated that such material would have a density of about eight under normal laboratory conditions. This means that the central regions of the Earth are occupied by material with a density close to that of iron or copper. It is usually inferred from this result that the core of the Earth is occupied by elements with atomic weights lying between 50 and 70. The core may however contain ten per cent. by mass of the heaviest elements without any disagreement with observation arising.

It will be seen from the above remarks that the evidence available concerning the distribution of the elements in the universe is very fragmentary. Nevertheless the astrophysical data are sufficient to bring out the important result that hydrogen is by far the most abundant element. This conclusion gives strong support to the view that it should be possible to trace back the history of the universe to a state in which hydrogen is the only element present. That is, it must be possible to show how the various elements have been synthesized from hydrogen. The present paper attempts to show how such a synthesis takes place. Stated more explicitly we suppose:

(i) Initially the only element present in the universe is hydrogen.

(ii) Helium is synthesized by thermonuclear reactions taking place in "normal" stars.

(iii) A further process occurs that synthesizes higher elements from hydrogen and helium. The elements produced are regarded as having a distribution similar to that found on the Earth.

The object of the present paper is to discuss (iii).

1.2. *Relation with previous investigations.*—The equations of statistical equilibrium between atomic nuclei have been discussed by a number of authors.† The most recent work is by Chandrasekhar and Henrich (referred to below as the C-H theory). The C-H theory deals with the relative abundance of the elements at temperatures between 8×10^9 and 6×10^9 °C. and at densities of order 10^7 gm. per cm.^{3}. Under these conditions only about one part in a million by mass is converted into elements of atomic weight greater than that of helium. The relative abundances of the elements are well reproduced in the range between *C* and *Cl*, but beyond atomic weight 40 the abundances given by the C-H theory become quite negligible. Thus the C-H theory fails to account for the existence of elements beyond argon. One of the main features of the present work is that the required abundances will be obtained over the whole of the periodic table. The chief points of difference between the present and previous discussion may be summarized as follows:—

(1) Account is taken of the degeneracy of the electrons at high densities.

* H. Jeffreys, The Earth, p. 220, Cambridge, 1929; *M.N.*, *Geophys. Suppl.* **4**, 62, 1937.

† T. E. Sterne, *M.N.*, **93**, 736, 1933; R. H. Fowler, Statistical Mechanics, p. 655, Cambridge, 1936; S. Chandrasekhar and L. R. Henrich, *Ap. J.*, **95**, 288, 1942.

(2) A large fraction of the material is converted into elements heavier than helium (as opposed to one part in a million in the C-H theory).

(3) The inclusion of electron degeneracy enables the required relative abundances to be obtained over the whole range of the periodic table without the temperature of the assembly being appreciably greater than 10^{10} °C.

In addition to these technical points the present work differs from the C-H theory on a number of questions of principle:

(*a*) It is not sufficient to find the values of ρ, T required for an assembly to contain the proper relative abundances of the elements. It is also necessary that ρ, T can be converted to low density and temperature (such as occurs on the Earth or in interstellar space) *without the relative abundances being changed.* This means that the mixture must remain frozen during some cooling-expansion process, for if statistical equilibrium were maintained throughout the process the relative abundances would undergo important changes.

(*b*) It is necessary to find the *place in the universe* where the required values of ρ, T occur.

In addition to previous investigations dealing with the synthesis of elements there are also a number of papers concerned with the properties of matter at high temperatures and densities that the writer has read with interest.*

1.3. *Procedure to be followed.*—The chief aims of the present paper may be summarized briefly:—

(1) To find values of ρ, T such that the equations of statistical equilibrium give the required abundances of the elements over the whole range of the periodic table.

(2) To show that a cooling-expansion process can be found such that the values of ρ, T satisfying (1) can be reduced to low density and temperature without the composition of the mixture being changed.

(3) To find a place in the universe where the values of ρ, T satisfying (1) must occur and to explain in a satisfactory way the origin of the cooling-expansion process.

There are two ways of discussing this programme. The indirect way is to construct the equations of statistical equilibrium and to search by trial and error until values of ρ, T are obtained that satisfy (1). The next step would be to use these values of ρ, T in a search for processes satisfying (2), (3) and (4). The second and more direct way is to discuss (3) and to find what values of ρ, T satisfy (2) also. Then having arrived at a pair of values of ρ, T (or possibly at more than one pair of values) we can use (1) and (2) as consistency checks on the theory. The latter procedure will be adopted in the following work. It is shown below that consideration of (2) and (3) leads to two possible ranges of values of ρ, T. One of these ranges yields the abundance of the elements of atomic weight up to about 80 (this is material described as case (1) in Section 8), whilst the other case yields the required abundances over the remainder of the periodic table (this material is described in Section 8 as case (2)).

An important by-product of the present investigation is a natural explanation of the difference between novae and supernovae. The theory of supernovae here given is similar in a number of respects to the neutrino-emission theory

* W. Baade and F. Zwicky, *Phys. Rev.*, **45**, 138, 1934 and **46**, 76, 1934; L. Landau, *Nature*, **141**, 333, 1938; F. Zwicky, *Phys. Rev.*, **55**, 726, 1939; F. Cernushi, *Phys. Rev.*, **56**, 450, 1939.

given by Gamow and Schoenberg.* The chief point of difference is that whereas in the Gamow-Schoenberg theory removal of energy occurs only through neutrino emission, in the present theory a more rapid rate of removal of energy arises from nuclear transformations.

2. *Some General Properties of Stars*

2.1. *Normal Stars.*—In the present section we shall consider stars in which nuclear transformations do *not* abstract thermal energy from stellar material (as will be seen in later sections important cases can occur where the absorption of energy by nuclear reactions becomes the dominant process in determining the evolution of a star). Under these conditions the only important way in which a star loses energy is by radiation from the surface. It follows from this, as will be proved at the end of the present subsection, that the star must satisfy the hydrostatic equation

$$\mathfrak{R} \frac{d}{dr}\left(\frac{\rho T}{\mu\beta}\right) = -\frac{GM(r)\rho}{r^2} \qquad (1)$$

to a very high degree of approximation, where $\mathfrak{R}$ is the gas constant; μ, ρ, T are the mean molecular weight, density and temperature of the stellar material; β is the ratio of the gas pressure $\mathfrak{R}\rho T/\mu$ to the total pressure $\mathfrak{R}\rho T/\mu + aT^4/3$, which includes the pressure $aT^4/3$ due to radiation, where a is Stefan's constant; G is the gravitational constant, r is a radial coordinate that measures distance from the centre of the star, and $M(r)$ is the mass of the star contained in a sphere of radius r.

Now when a star satisfies (1) either accurately or approximately the central temperature T_c cannot differ by a factor of much more than two from the value †

$$\frac{G\mu\beta_c M}{3\mathfrak{R}R}, \qquad (2)$$

where M is the total mass of the star, β_c is the value of β at the centre, and R is the radius of the star (more accurately R should be taken such that $M(R) \simeq M/2$, because it is possible for a star to possess an outer envelope of large radius but containing only a small fraction of the mass if μ is appreciably less in the envelope than in the interior ‡). The temperature of the surface of the star is small compared with (2) and there must be an outward temperature gradient of order

$$\frac{G\mu\beta_c M}{3\mathfrak{R}R^2}$$

in the star. This temperature gradient will result in there being an outward flux of energy from inside the star to the surface. The magnitude of this energy flux follows immediately from the well known equation of radiative equilibrium and is of order

$$\frac{16\pi caR}{3\kappa\bar{\rho}^2(1-X^2)}\left(\frac{G\mu\beta_c M}{3\mathfrak{R}R}\right)^{7\cdot5}, \qquad (3)$$

where κ is the coefficient of opacity, c is the velocity of light, X is the fraction

* G. Gamow and M. Schoenberg, *Phys. Rev.*, **59**, 539, 1941 ; G. Gamow, *Phys. Rev.*, **59**, 617, 1941 and **65**, 20, 1944.

† A. S. Eddington, I.C.S., p. 92, Cambridge, 1930.

‡ F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 218, 1942.

by mass of hydrogen in the star, and $\bar{\rho}$ the mean density of material in the star is given by

$$\frac{4}{3}\pi R^3\bar{\rho}=M. \tag{4}$$

Substituting for $\bar{\rho}$ in (3) leads to the following expression for the energy flux:

$$\frac{3ca}{\pi\kappa(1-X^2)}\cdot\left(\frac{G\mu\beta_c}{3\mathfrak{R}}\right)^{7\cdot5}.M^{5\cdot5}R^{-0\cdot5}. \tag{5}$$

The expression (5) not only gives the outward flux of energy in the star but also gives the rate of radiation from the surface. Thus, if the rate of radiation from the surface is greater than (5), the surface temperature must fall, since the loss of energy at the surface cannot be compensated by the flux of energy from inside the star. Similarly, if the radiation from the surface is less than (5), the surface temperature must rise, since the supply of energy from inside the star exceeds the loss by radiation.

If thermonuclear reactions taking place in the star generate sufficient energy to compensate exactly for the loss of energy at the surface, then the star remains in equilibrium. If the generation of energy exceeds the rate of loss then the star will expand. As will be seen from (2) the temperature in the energy-producing regions falls with the expansion, and this has the effect of rapidly reducing the rate of production of energy (the carbon nitrogen cycle gives an energy production approximately proportional to ρ^2T^{18}, whilst direct synthesis helium from hydrogen gives an energy generation approximately proportional to $\rho^2T^{4\cdot5}$). Thus the star expands until the energy production falls to a value equal to the loss at the surface. If, on the other hand, the generation of energy due to the synthesis of helium from hydrogen is less than the loss at the surface, then the star will contract and the temperature will rise in accordance with (2). Thus in this case, provided the quantity of hydrogen present in the energy-producing regions is not negligible, the star will contract until energy production becomes equal to the loss at the surface (the discussion given in 2.2 shows that the present proviso is very important). We may conclude that stars containing an appreciable amount of hydrogen will adjust their dimensions (and hence their central temperatures) in such a way that the energy generation is just sufficient to balance the loss at the surface, which is given approximately by (5).

Now the generation of energy by the carbon nitrogen cycle for example (similar remarks apply to direct synthesis of helium from hydrogen) can be written in the approximate form *

$$4\pi E\int_0^R \rho^2T^{\eta}r^2\,dr, \tag{6}$$

provided the central temperature is not too far from the equilibrium value, where η varies nearly as $M^{-1/12}$ and is about 18 for the Sun, and E is an energy production constant. The integral (6) can be written as

$$\frac{4}{3}\pi E\lambda R^3\bar{\rho}^2T_c^{\eta},$$

where λ is an averaging constant that has effectively the same value from one star

* F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 177, 1942.

to another. Substituting for $\bar{\rho}$ from (4) and for T_c from (2) gives an energy production

$$\frac{3E\lambda}{4\pi}\left(\frac{G\mu\beta_c}{3\mathfrak{R}}\right)^{\eta}\frac{M^{\eta+2}}{R^{\eta+3}}. \tag{7}$$

For equilibrium conditions the expressions (5) and (7) are equal and R is given approximately by

$$\left\{\frac{E\lambda\kappa(1-X^2)}{4ac}\right\}^{\frac{2}{2\eta+5}}\cdot\left(\frac{G\mu\beta_c}{3\mathfrak{R}}\right)^{\frac{2\eta-15}{2\eta+5}}\cdot M^{\frac{2\eta-7}{2\eta+5}}. \tag{8}$$

Thus, since it can be shown that β_c can be expressed in terms of μ and M, it follows that R can be calculated * explicitly in terms of μ and M. The value of R so determined may be substituted in (5) to give the rate of emission L at the surface in terms of μ and M. Thus *both* R and L can be calculated explicitly in terms of μ and M. These remarks refer to stars in equilibrium. Such stars are *normal* in the sense used in the Introduction.

In a refined discussion of the structure of stars the approximate expressions (5) and (7) have to be replaced by accurate values (the expressions (5) and (7) are approximate in the numerical constants multiplying them and not in the dependence on M, R and μ). † For the present purposes the above discussion is preferable because a number of points are brought out that tend to be obscured in a more detailed theory. Moreover the present investigation shows what results must of necessity turn up in a thorough-going analysis. For example, it is seen from (8) that $R \propto E^{2/(2\eta+5)}$ and from (5) that $L \propto E^{-1/(2\eta+5)}$, so that both R and L are very insensitive to the value of the energy production constant. Furthermore we see that this result is inherent in the problem and cannot be removed by more detailed calculations.

Apart from a numerical factor the expression (5) is identical with a well-known formula due to Eddington.

At this stage we may dispose of the asumption made at the beginning of the present subsection. That is, we have to show that a collapsing star which *only loses energy through radiation at the surface* must satisfy (1) to a high degree of approximation. The total thermal energy contributed by the material of a star is of order $\mathfrak{R}MT_c$. Now the star cannot contract appreciably until an amount of energy has been radiated at the surface that is of the same order as $\mathfrak{R}MT_c$. Thus the time required for appreciable contraction to occur is of order $\mathfrak{R}MT_c/L$. To obtain an idea of the magnitude of this quantity put $M=M_\odot=2\times 10^{33}$ gm., $T_c=2\times 10^7\,^\circ$C., $L=L_\odot=4\times 10^{33}$ ergs per sec. Then the time required for appreciable contraction to occur is of order $2{\cdot}8\times 10^7$ years. This time may be compared with the time that would be required for collapse if (1) were not satisfied to a good approximation. The latter time is easily seen to be of order $\sqrt{(R^3/GM)}$. For $R=R_\odot=7\times 10^{10}$ cm. and $M=M_\odot$, $\sqrt{(R^3/GM)}$ is about $1{\cdot}6\times 10^3$ sec. Although for different M, T_c, R the numerical values will differ from those given here it is clear that in all cases

$$\frac{\mathfrak{R}MT_c}{L} \gg \sqrt{\left(\frac{R^3}{GM}\right)}.$$

* F. Hoyle and R. A. Lyttleton, *loc. cit.*

† F. Hoyle and R. A. Lyttleton, *loc. cit.*

This proves the required result that (1) must be satisfied to a good approximation when the only loss of energy is by radiation at the surface.

2.2. *Collapsing Stars.*—In this subsection we consider a collapsing star in which the rate of generation of energy is less than the rate of radiation from the surface. Two different cases may be distinguished:

(1) The star contains an appreciable fraction by mass of hydrogen but the internal temperature is too low to give sufficient energy generation for equilibrium. This case occurs during the condensation of a star from interstellar material, and the collapse continues until the temperature rises to a sufficiently high value for equilibrium to occur.

(2) The generation of energy by synthesis of helium from hydrogen may cease, due to the exhaustion of the supply of hydrogen in the energy-producing regions (hydrogen may still be present in the atmosphere). This will occur in a period of about 10^8 years (compared with an age of the universe of about 10^{10} years) for stars of large mass and luminosity unless the hydrogen is replenished by accretion of interstellar hydrogen. Thus, if accretion ceases to be important * for such a star, the supply of hydrogen will become exhausted in a period of time that is short compared with the age of the universe. The star will then enter a period of contraction. As the radius decreases the central temperature will increase in accordance with (2). The rising temperature will remove the last vestiges of hydrogen present in the inner parts of the star and the star if sufficiently massive must collapse until rotational instability occurs.† The condition on the mass is that it must appreciably ‡ exceed Chandrasekhar's limit $5{\cdot}75 M_{\odot}/\mu_e^2$. Sufficient material is thrown off to infinity during the instability process for the mass of the remaining stellar nucleus to be reduced to a value of the order of, or less than, Chandrasekhar's limit.

From the point of view of the present paper two different cases may be distinguished. These two cases may be examined by discussing the instability process in further detail.

(i) The temperature rises sufficiently high for nuclear reactions to become important *before* the onset of rotational instability (a temperature of about 4×10^9 °C. is required).

(ii) A state of rotational instability is reached before nuclear reactions become important.

Owing to the conservation of angular momentum, any initial rotation possessed by a star increases as the star contracts. During contraction the rotational velocity is proportional to the reciprocal of the radius of the star. Thus the centrifugal force acting on the stellar material increases as the inverse cube of the radius. It follows that, since the gravitational force increases only as the inverse square of the radius, the ratio of centrifugal force to gravitational force must increase inversely as the radius. When the centrifugal force becomes comparable with the gravitational force, instability occurs. If we regard equality

* F. Hoyle, *M.N.*, **105**, 363, 1945.

† F. Hoyle, *M.N.*, in course of publication.

‡ The quantity μ_e is defined somewhat differently from μ. Thus whereas μ is given by the product of the mass of the hydrogen atom and the number of particles per gram of material, the quality μ_e is defined as the product of the mass of the hydrogen atom and the number of free electrons per gram of material. For the stars considered in the present paper μ_e is always close to 2.

of these two forces as an approximate criterion for rotational instability, then we can estimate the degree of contraction required to produce instability for various values of the initial equatorial rotational velocity. For example, consider a star of mass $10M_\odot$ that exhausts its supply of hydrogen and enters the period of contraction discussed above. Before contraction begins, the values of ρ, T at the centre of the star would be about 3 gm. per cm.³ and $3 \cdot 7 \times 10^7$ °C., whilst the radius would be about $3 \cdot 5 \times 10^{11}$ cm. To a sufficient approximation we may suppose that the star contracts through a series of homologous configurations, so that the central values of ρ, T are proportional to the inverse cube and the inverse first power respectively of the radius. The following table gives the various quantities that are of interest for a set of values of the initial rotational velocity:—

TABLE I

The Effect of the Initial Rotational Velocity on Collapsing Stars

Mass $= 10M_\odot$. Initial radius $= 3 \cdot 5 \times 10^{11}$ cm.

Initial rotational velocity (km. per sec.).	1	5	10	15	20	40	100
Factor by which radius must be reduced to make centrifugal and gravitational forces equal.	$3 \cdot 82 \times 10^5$	$1 \cdot 53 \times 10^4$	$3 \cdot 82 \times 10^3$	$1 \cdot 70 \times 10^3$	$9 \cdot 52 \times 10^2$	$2 \cdot 38 \times 10^2$	$3 \cdot 82 \times 10$
Initial central temperature in °C.	$3 \cdot 5 \times 10^7$	$3 \cdot 5 \times 10^7$	$3 \cdot 5 \times 10^7$	$3 \cdot 5 \times 10^7$	$3 \cdot 5 \times 10^7$	$3 \cdot 5 \times 10^7$	$3 \cdot 5 \times 10^7$
Central temperature at time of rotational instability in °C.	$>4 \times 10^9$	$>4 \times 10^9$	$>4 \times 10^9$	$>4 \times 10^9$	$>4 \times 10^9$	$>4 \times 10^9$	$1 \cdot 34 \times 10^9$
Initial central density in gm. per cm.³.	3	3	3	3	3	3	3
Central density at onset of rotational instability in gm. per cm.³.	$1 \cdot 67 \times 10^{17}$	$1 \cdot 07 \times 10^{13}$	$1 \cdot 67 \times 10^{11}$	$1 \cdot 47 \times 10^{10}$	$2 \cdot 59 \times 10^9$	$4 \cdot 07 \times 10^7$	$1 \cdot 67 \times 10^5$

The values given in Table I are necessarily approximate (in particular relativity effects are neglected in the first set of values), but even so they show clearly that unless the initial rotational velocity of the star is of the order of, or greater than, 100 km. per sec. the temperature will rise to 4×10^9 °C. before rotational instability occurs. Now at a temperature of 4×10^9 °C. nuclear reactions become important and we have case (i). Indeed it follows from Table I that, if the initial rotational velocity is of the order of, or less than, 40 km. per sec., we have case (i), whereas, if the initial rotational velocity is of the order of, or greater than, 100 km. per sec., we have case (ii). Thus, since the initial rotational velocity will vary from one star to another, it follows that some stars will satisfy case (i) and others case (ii). In the present paper we are only concerned with stars in which the initial rotational velocity is small enough for case (i) to apply. The range of values of the initial rotational velocity used in the following work is roughly 10 to 40 km. per sec.

The large densities appearing in the above table suggest that electron degeneracy may arise during the contraction of the star, and that the contribution of the pressure of the degenerate electrons may become important. As will be seen in a later section, this is the case, but only *after* nuclear reactions have become important. Before nuclear reactions become important, the density increases at the same rate as the cube of the temperature, so that the ratio of gas pressure to radiation pressure remains constant during this phase of the contraction. Now, as Chandrasekhar has shown*, pressure degeneracy does not become important unless β_c is greater than a value close to 0·908. From the known relation† between β_c and M, it can be shown that β_c will not exceed 0·9 unless $M\mu_e^2/M_\odot < 5{\cdot}75$. In the following discussion we shall only consider stars satisfying $M\mu_e^2/M_\odot > 5{\cdot}75$, so that degeneracy does not arise before nuclear reactions become important. If a star undergoes appreciable contraction after the temperature has risen to 4×10^9 °C., then degeneracy will always occur. This is due to nuclear reactions which prevent the temperature from rising appreciably above 10^{10} °C. At this stage β_c increases, because the gas pressure continues to increase on account of the increasing density but radiation pressure remains effectively constant. If sufficient contraction occurs, β_c will rise above 0·908 and the stellar material will become degenerate.

2.3.—The remarks of 2.2 provide a suitable basis for a theory of the synthesis of the elements. It has been shown that the temperatures arising in collapsing stars that have exhausted their supply of hydrogen become sufficiently high for statistical equilibrium between atomic nuclei to occur provided

(1) The mass of the star appreciably exceeds Chandrasekhar's limit.

(2) The initial rotational velocity before collapse begins is less than about 40 km. per sec.

Thus our first object of finding a place in the universe where the elements may be synthesized has been achieved. Furthermore, a cooling-expansion process of the type discussed in the Introduction is also available in the process of rotational instability. This serves not only to reduce the density and temperature but to distribute the elements in interstellar space.

In view of the importance of the processes discussed in 2.2 it is desirable to seek independent confirmation of these results. It is therefore fortunate that observational evidence can be cited in support of the starting-point of the present investigations. Thus what appears to be an excellent example of the break up of a rotationally unstable star has recently been investigated in detail by Baade and Minkowski.‡ In this particular case it is possible to observe directly the nebulous material that was thrown off by the supernova of 1054 A.D. The mass of this material is found to be of order 15 $M_\odot$ so that the star must have thrown off an appreciable proportion of its original mass. The estimate given by Minkowski for the radius of the remaining stellar nucleus confirms that the material in the nucleus is very dense, and thereby provides support for the view that the outburst occurred in a collapsing star whose supply of hydrogen had become exhausted. Thus, if we take the mass in the nucleus as $M_\odot$, which is probably an underestimate, together with Minkowski's estimate of about 10^9 cm. for the radius, the present mean density is about 5×10^5 gm. per cm.^{3}. Allowing for a central condensation of about 20 (the central density in

* S. Chandrasekhar, Introduction to the Study of Stellar Structure, p. 434, Chicago, 1938.

† F. Hoyle and R. A. Lyttleton, *loc. cit.*

‡ W. Baade, *Ap. J.*, 96, 188, 1942 ; R. Minkowski, *Ap. J.*, 96, 199, 1942.

a star is about 20 times the mean density), this gives a present central density in the nucleus of about 10^7 gm. per cm.^{3}. We may note further that the central density before outburst would be unlikely to be less than 10^7 gm. per cm.^{3}. Thus, taking 16 $M_\odot$ as a rough estimate of the initial mass, it follows that the radius at outburst could not be much greater than $2 \cdot 5 \times 10^9$ cm. If this value of the radius is used in (2), together with $\mu\beta_c = 1$, an estimate of $3 \cdot 4 \times 10^9$ °C. is obtained for the central temperature before outburst. This value is just of the order that is of interest in the present problem. Accordingly it is seen that there is not only theoretical evidence but also observational data in favour of the model used in the following sections of the present paper.

3. *The Equations of Statistical Equilibrium*

The next step is to construct the equations of statistical equilibrium between atomic nuclei (these equations applying at temperatures exceeding 4×10^9 °C.). Write n_P, n_N, n_A^Z for the number of free protons per cm.^{3}, the number of free neutrons per cm.^{3}, and the number of atomic nuclei with atomic weight A and charge Z per cm.^{3}, respectively. Then under conditions of statistical equilibrium n_A^Z can be expressed in terms of the three quantities n_P, n_N and T. These equations are well known and will be quoted here.* In Fowler's notation the required equations are

$$\left.\begin{aligned} Vn_P &= \xi \frac{\partial}{\partial \xi}\left\{\sum_q \ln\left(1 + \xi e^{-\epsilon_q/kT}\right)\right\}, \\ Vn_N &= \zeta \frac{\partial}{\partial \zeta}\left\{\sum_r \ln\left(1 + \zeta e^{-\epsilon_r/kT}\right)\right\}, \\ Vn_A^Z &= \chi \frac{\partial}{\partial \chi}\left\{\sum_s \pm \ln\left(1 \pm \chi e^{-\epsilon_s/kT}\right)\right\}, \end{aligned}\right\} \qquad (9)$$

where V is the volume occupied by the assembly and ϵ_q, ϵ_r, ϵ_s represent the energy states (energy of translation together with internal excitation) of the proton, neutron and nucleus of atomic weight A and charge Z, respectively. The $\pm$ sign depends on A; if A is odd the nuclei obey Fermi-Dirac statistics and the plus sign must be taken, whilst if A is even the nuclei obey Einstein-Bose statistics and the minus sign must be used. The parameters ξ, ζ, χ appear in using the theorem of steepest descents and they satisfy the relations

$$\left.\begin{aligned} \chi &= \xi^Z \zeta^{(A-Z)} \exp\left(-Q_A^Z/kT\right), \\ Q_A^Z &= c^2\{m_A^Z - Zm_P - (A-Z)m_N\}, \end{aligned}\right\} \qquad (10)$$

m_P, m_N, m_A^Z being the masses of the proton, neutron and nucleus of weight A and charge Z, respectively. There is a set of equations of the form (9) and (10) for each pair of values of A, Z.

The above equations can be simplified in two ways. First, to the degree of accuracy aimed at in the present paper the contribution to ϵ_s of the excited states of the nuclei can be neglected (the errors arising from this step are not greater than a factor of about 10^2 in the relative abundances of the nuclei. This is small compared with the factors that arise from other considerations). Second,

* R. H. Fowler, *op. cit.*, p. 655. The symbol " ln " denotes the logarithm to the base e, and the symbol " log " will be used to denote logarithms to the base 10.

even at the highest densities discussed below ($\simeq 10^{11}$ gm. per cm.^{3}), the parameters ξ, ζ, χ are sufficiently small compared with unity for it to be possible to use "classical" statistics for the protons, neutrons and heavy nuclei. These simplifications enable the sums in (9) to be written as *

$$\left.\begin{aligned} \sum_q \ln(1+\xi e^{-\epsilon_q/kT}) &= \xi V(2\pi m_P kT)^{\frac{3}{2}}/h^3, \\ \sum_r \ln(1+\zeta e^{-\epsilon_r/kT}) &= \zeta V(2\pi m_N kT)^{\frac{3}{2}}/h^3, \\ \pm\sum_s \ln(1\pm\chi e^{-\epsilon_s/kT}) &= \chi V(2\pi m_A^Z kT)^{\frac{3}{2}}/h^3. \end{aligned}\right\} \tag{11}$$

Equations (11) together with (10) give

$$\ln n_A^Z = Z\ln n_P + (A-Z)\ln n_N + (A-1)\ln\{h^3/(2\pi kT)^{\frac{3}{2}}\} - \frac{Q_A^Z}{kT} - \frac{3}{2}Z\ln m_P - \frac{3(A-Z)}{2}\ln m_N + \frac{3}{2}\ln m_A^Z. \tag{12}$$

This equation has been expressed in a convenient numerical form by Chandrasekhar and Henrich †, who use 10^9 °C. as the unit of T, and the millimass unit ($10^{-3}\, m_P$) as the unit of Q_A^Z. With these units, and putting $m_P = m_N$, equation (12) becomes

$$\log n_A^Z = 34{\cdot}08 + \frac{3}{2}\log T + \frac{3}{2}\log A + \frac{4{\cdot}73}{T}\,.\,Q_A^Z + A\left(\log n_N - 34{\cdot}08 - \frac{3}{2}\log T\right) - Z\log\left(\frac{n_N}{n_P}\right). \tag{13}$$

Equation (13) gives one of the basic equations of the present problem. The remaining equations are rather less straightforward in their derivation and will be treated in greater detail. A relation can be derived between n_N/n_P and n_e, where n_e is the electron density. This relation is due to interchange between an electron and a proton on the one hand and a neutron on the other hand. The interchange arises from β-processes taking place among the nuclei. The neutrinos and antineutrinos (according to the usual formulation of the properties of these particles) emitted in the β-processes will escape from the star and will remove energy in the manner discussed by Gamow and Schoenberg.‡ Provided the interchanges are sufficiently frequent (we shall assume this to be the case), statistical equilibrium will be set up, which will be similar in character to the case of a simple dissociating gas. In the latter problem we have atoms of type A_1 combining with atoms of type A_2 to form molecules $A_1 A_2$, together with the reverse reaction in which $A_1 A_2$ dissociates into the separate atoms, A_1 and A_2. This can be represented symbolically as

$$A_1 + A_2 \rightleftarrows A_1 A_2.$$

The solution of this problem is known § and can be taken over to the proton, neutron, electron case by regarding the neutron as the "molecule" and the proton and electron as the "atoms". The equations determining the equilibrium are

* R. H. Fowler, *op. cit.*, p. 655.
† Chandrasekhar and Henrich, *op. cit.*
‡ Gamow and Schoenberg, *loc. cit.*
§ Fowler, *op. cit.*, p. 157.

the first two equations of (9) together with

$$\left.\begin{aligned} Vn_e &= \lambda \frac{\partial}{\partial \lambda}\left\{\sum_t \ln\left(1+\lambda e^{-\epsilon_t/kt}\right)\right\}, \\ \zeta &= \lambda \xi \exp(-\mathcal{J}/kT), \\ \mathcal{J} &= c^2(m_N - m_P - m_e), \end{aligned}\right\} \tag{14}$$

where ϵ_t represents the energy levels of the electron and m_e is the electron mass. Putting ζ, ξ equal to the values given by (9) and (11) gives

$$\frac{n_N}{n_P} = \lambda \left(\frac{m_N}{m_P}\right)^{\frac{3}{2}} \exp\left(-\frac{\mathcal{J}}{kT}\right) \simeq \lambda e^{-\mathcal{J}/kT}. \tag{15}$$

The first of equations (14) determines λ in terms of n_e. This equation is comparatively easy to solve for both the cases $\lambda \ll 1$ and $\lambda \gg 1$, but is rather awkward when $\lambda \simeq 1$. Throughout the following work the form of solution in which $\lambda \gg 1$ will be used. This is necessary at the highest densities considered (of order 10^{11} gm. per cm.^{3}) but is only a moderate approximation at the lowest densities (of order 10^7 gm. per cm.^{3}) treated in the work of later sections. This procedure is however fully justified by the circumstances that in calculations of the relative abundances of the elements the terms depending on λ are unimportant at the lowest densities (10^7 gm. per cm.^{3}) but are very important at the highest densities (10^{11} gm. per cm.^{3}). Thus our approximation is very good in the range of density where the terms in λ matter, whilst the approximation is only moderate for the range of density in which the terms in λ are unimportant. When $\lambda \gg 1$ and account is taken of the relativistic variation of mass of the electrons in forming the energy levels ϵ_t, the first of equations (14) can be solved to give*

$$\left.\begin{aligned} \ln \lambda &= m_e c^2 \sqrt{(1+x^2)}/kT, \\ x &= (n_e/5\cdot 87 \,.\, 10^{29})^{\frac{1}{3}}, \end{aligned}\right\} \tag{16}$$

the temperature being here measured in °C. Equations (15) and (16) give

$$\left.\begin{aligned} \ln\left(\frac{n_N}{n_P}\right) &= \frac{m_e c^2}{kT}\left\{\sqrt{(1+x^2)} - \frac{\mathcal{J}}{m_e c^2}\right\} = \frac{m_e c^2 y}{kT}, \\ \text{where} \quad y &= \sqrt{(1+x^2)} - \mathcal{J}/m_e c^2 = \sqrt{(1+x)^2} - 0\cdot 51. \end{aligned}\right\} \tag{17}$$

If we again measure T in units of 10^9 °C., the first of equations (17) becomes

$$\log \frac{n_N}{n_P} = \frac{4\cdot 73}{T} \,.\, 0\cdot 543 y. \tag{18}$$

Substituting in (13) gives

$$\log n_A^Z = 34\cdot 08 + \frac{3}{2}\log T + \frac{3}{2}\log A + \frac{4\cdot 73}{T}(Q_A^Z - 0\cdot 543 yZ)$$
$$+ A(\log n_N - 34\cdot 08 - \frac{3}{2}\log T). \tag{19}$$

The second of equations (16), equations (18) and (19) give the complete scheme of statistical equations in the present problem.

A more useful form of the equation determining the parameter x can be

* S. Chandraskehar, *op. cit.*, p. 393 ; R. H. Fowler, *op. cit.*, p. 652.

obtained by expressing n_e in terms of the density of the material. If we write ρ' for the density excluding the contribution of the free neutrons (in the following discussion the free protons never make an appreciable contribution to the density, but at the highest densities considered below it turns out that the free neutrons do make an appreciable contribution), then to a sufficient approximation we can write

$$n_e \simeq \frac{\rho'}{2m_P} \tag{20}$$

for material not containing hydrogen. This equation follows from the consideration that the assembly must be effectively electrically neutral. This requires that n_e shall be equal to the total positive charge density of the atomic nuclei, which is close to $\rho'/2m_P$ together with a contribution equal to the density of the positrons present in the assembly. The latter term can be calculated from statistical mechanics * and it turns out that its value can be neglected in comparison with $\rho'/2m_P$. Except when ρ' is higher than 10^{10} gm. per cm.^{3}, it is sufficient to put ρ' equal to the total density ρ. The equation for x becomes

$$\rho' = 1{\cdot}98 \times 10^6 x^3. \tag{21}$$

Equation (21) enables x and therefore y to be determined in terms of ρ'. Thus, since Q_A^Z is a known quantity, it follows from (19) that n_A^Z is determined in terms of ρ', T and n_N.

4. *The Helium Zone*

4.1. It would now be possible to return to the physical model discussed in Section 2 and to apply the statistical equations of Section 3. Before doing this, however, it is more convenient to discuss the further properties of these equations and to return to the physical model at a later stage. The two questions that will be considered in the present section and in Section 5 respectively are:

(1) If we regard T, ρ' as Cartesian coordinates, then in the T, ρ'-plane a curve can be drawn with the following important property:—Material at values of T, ρ' represented by any point on one side of the curve is composed almost entirely of helium, whereas material at any point, T, ρ' on the other side of the curve is composed almost entirely of heavy elements, the main mass of the elements in the latter case having atomic weight greater than 50.

(2) Throughout Section 2 it was assumed that nuclear reactions take place sufficiently rapidly when $T > 4 \times 10^9$ °C., $\rho' > 10^7$ gm. per cm.^{3} for the statistical equations to be applicable, whereas when $T < 4 \times 10^9$ °C. it was supposed that the mixture remains frozen. This question will be discussed in detail in Section 5.

It will now be shown when statistical equilibrium occurs that, for a given density ρ', there is a closely determinable value of T in the neighbourhood of which the material changes over from being almost entirely composed of helium to being almost entirely composed of heavy elements. It has to be shown further that in the latter case the most abundant element has atomic weight greater than 50. It is most convenient to assume the latter result to begin with, and then to prove it at the end of the discussion.

Equation (19) can be used to express n_4^2 in terms of y (i.e. ρ'), T, and n_N.

* R. H. Fowler, *op. cit.*, p. 653, equation (1830).

Thus we have

$$\log n_4^2 = 34{\cdot}08 + \frac{3}{2}\log T + \frac{3}{2}\log 4 + \frac{4{\cdot}73}{T}(Q_4^2 - 1{\cdot}086y)$$
$$+ 4(\log n_N - 34{\cdot}08 - \frac{3}{2}\log T). \quad (22)$$

Equations (19) and (22) enable n_N to be eliminated and give the following relation between n_A^Z and n_4^2:—

$$\frac{\log(An_A^Z) - 34{\cdot}08 - 3/2\log T - 5/2\log A}{A} = \frac{\log 4n_4^2 - 34{\cdot}08 - 3/2\log T - 5/2\log 4}{4}$$
$$+ \frac{4{\cdot}73}{T}\left\{\frac{Q_A^Z}{A} - \frac{Q_4^2}{4} + 0{\cdot}543\,y\left(\frac{1}{2} - \frac{Z}{A}\right)\right\}. \quad (23)$$

Consider now the orders of magnitude of the three terms in equation (23).

(i) *Second term on right-hand side of equation* (23). The quantity Q_A^Z/A is the packing fraction of atoms of atomic weight A and charge Z. The values of the various quantities involved are shown in Table II for a number of elements, using experimental values for the packing fraction.*

TABLE II

	He^4	O^{16}	Si^{28}	Fe^{56}	Cu^{63}	Kr^{82}	Sn^{118}	Pb^{208}
Q_A^Z/A	7·57	8·54	9·01	9·27	9·27	9·22	9·12	8·35
Z/A	0·5	0·5	0·5	0·46	0·46	0·44	0·42	0·40
$Q_A^Z/A - Q_4^2/4$	0·0	0·97	1·44	1·70	1·70	1·65	1·55	0·78
$Q_A^Z/A - Q_4^2/4 + 0{\cdot}543y(1/2 - Z/A)$	0·0	0·97	1·44	1·70 $+0{\cdot}02y$	1·70 $+0{\cdot}02y$	1·65 $+0{\cdot}034y$	...	...

These values taken together with the following values of y for various ρ' show that the second term on the right-hand side of (23) is of order unity when $A > 16$ and $4 < T < 10$:—

TABLE III

ρ' (gm. per cm.^{3})	10^7	10^8	10^9	10^{10}	10^{11}
x	1·72	3·70	7·96	17·2	37·0
y	1·48	3·32	7·51	16·6	36·5

It will be remembered that the form of the term in y arises from the formulae for relativistically degenerate electrons. The values given in Tables II and III confirm the statement made in Section 3 that the term in y only becomes important when the use of these formulae gives a good approximation for the electron distribution (the condition for the approximation to be a good one is that $x \gg 1$).

(ii) *First term on the right-hand side of* (23).—If all the material were in the form of helium, then

$$\log 4n_4^2 = \log\left(\frac{\rho'}{m_P}\right) = \log\rho' + 23{\cdot}78,$$

whilst if other elements are present

$$\log 4n_4^2 < \log\rho' + 23{\cdot}78.$$

* O. Hahn, S. Flugge and J. Mattauch, *Phys. Z.*, 1941. The value used here for Kr^{82} is less than the value given by these authors. The present value has been chosen to fit smoothly into a maximum of the packing fraction between 60 and 70.